

Reciprocity and unitarity in scattering from a non-Hermitian complex PT-symmetric potential

Zafar Ahmed*

Nuclear Physics Division, Bhabha Atomic Research Centre, Mumbai 400 085, India

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Abstract

In non-relativistic quantum scattering, Hermiticity is necessary for both reciprocity and unitarity. Reciprocity means that both reflectivity (R) and transmitivity (T) are insensitive to the direction of incidence of a wave (particle) at a scatterer from left/right. Unitarity means that $R+T=1$. In scattering from non-Hermitian PT-symmetric structures the (left/right) handedness (non-reciprocity) of reflectivity is known to be essential and unitarity remains elusive so far. Here we present a surprising occurrence of both reciprocity and unitarity in scattering from a complex PT-symmetric potential. In special cases, we show that this potential can even become invisible ($R=0, T=1$) from both left and right sides. We also find that this optical potential can give rise to a perfect transmission ($T=1$) this time without both unitarity and reciprocity (of reflectivity).

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*Electronic address: zahmed@barc.gov.in

In non-relativistic quantum mechanics Hermiticity is the necessary condition for a Hamiltonian to have: real discrete spectrum, and both unitarity and reciprocity in scattering. Reciprocity means that both reflectivity (R) and transmittivity (T) are insensitive to the direction of incidence of a wave (particle) at a scatterer from left/right. Unitarity in scattering means that $R + T = 1$. In various branches of physics the complex optical potentials have been in use since a long time to account for the absorption of the incident flux in to unknown channels. Consequently, non-Hermiticity is synonymous to absorption or emission of flux. In this kind of scattering the unitarity is broken as the probability of reflection (R) and transmission (T) do not add to 1 and one instead has $R + T + A = 1$ where A is the probability of absorption.

About thirteen years ago Bender and Boettcher [1,2] conjectured that the eigenspectrum of a non-Hermitian complex potential in a parametric regime was discrete and real. This potential was PT-symmetric [invariant under Parity ($x \rightarrow -x$) and Time-reversal ($i \rightarrow -i$)]. Also this potential was not amenable to exact analytic solutions so it required special methods to prove the reality of its spectrum [3]. Their conjecture has initiated a debate: ‘Must a Hamiltonian be Hermitian?’ [2] and it has inspired a large body of investigations leading to the extension of the quantum mechanics in complex domain [1-20,22-25,28].

For the scattering from a complex non-Hermitian potential it has been possible to prove [4] that if non-Hermitian complex potential is spatially asymmetric the reflectivity (R) shows handedness $R_{left} \neq R_{right}$ whereas transmittivity (T) remains invariant to the direction of the incidence of the particle from left or right. The complex PT-symmetric potentials being spatially anti-symmetric are automatically entitled to this handedness [5-17]. This contrasting feature of scattering from that of reciprocity in Hermitian case perhaps may have discouraged one to look for unitarity in the scattering from complex PT-symmetric potentials. Nevertheless, various works [4-17] normally display non-unitarity in scattering from complex PT-symmetric potentials.

There has been a very impressive progress in the investigations of the scattering from a complex PT-symmetric potential. In some PT-symmetric structures the absence unitarity has been marked with new pseudo-unitarity conditions such as $T - 1 = \pm \sqrt{R_{left} R_{right}}, \frac{R_{left} + R_{right}}{2} + T = 1$ [9]. The concepts like spectral singularity [11-14] and invisibility [15,17] have been well developed both theoretically and experimentally. For the spectral singularity one looks for positive energies where there are very large (infinite) [11]

peaks in both $R(k)$ and $T(k)$. The instance where $R(k) = 0$ and $T(k) = 1$ is called unidirectional invisibility [15-17] in both complex-non-Hermitian and complex PT-symmetric potentials. Notice that this invisibility is direction dependent either from the left or from the right. This is the consequence of the handedness of reflectivity in these potentials.

The ghost of non-Hermiticity has already been busted in PT-symmetric domain, new features such as spectral singularity [11-14] and invisibility [15-17] of such potentials are being investigated. Novel optical devices and materials have been engineered to realize wave propagation through a complex PT-symmetric medium [9,15,16,18-20]. In this scenario of scattering from a complex PT-symmetric potential we present surprising parametric regimes in the well known complex PT-symmetric Scarf II potential wherein we observe reciprocity, unitarity, invisibility (from both left and right) and perfect transmission (with non-unitarity).

The scattering from Scarf II potential is well studied wherein the reflection and transmission amplitudes have been well worked out [5,21]. The non-Hermitian complex PT-symmetric version of the Scarf II potential has been very useful in the investigations of complex PT-symmetric potentials in various ways [5,7,13,14,22,23]. We would like to write the complex Scarf II potential as

$$V(x) = -(B^2 + A^2 + A)\text{sech}^2 x + iB(2A + 1)\tanh x \text{sech} x \quad (1)$$

which is known to have real discrete spectrum [5,22]. This potential in another parametric form displays phase-transition [23] of real discrete eigenvalues to complex conjugate pairs about a critical value of a parameter when the PT-symmetry breaks down [1,2]. Let $2\mu = 1 = \hbar^2$ and $k = \sqrt{E}$, where E is the energy. Following [5,21], we can write the transmission amplitude for (1) as

$$t_{A,B}(k) = \frac{\Gamma[-A - ik]\Gamma[1 + A - ik]\Gamma[\frac{1}{2} + B - ik]\Gamma[\frac{1}{2} - B - ik]}{\Gamma[-ik]\Gamma[1 - ik]\Gamma^2[\frac{1}{2} - ik]}, \quad (2)$$

$$r_{A,B}(k) = t_{A,B}(k)i \left[\frac{\cos \pi A \sin \pi B}{\cosh \pi k} + \frac{\sin \pi A \cos \pi B}{\sinh \pi k} \right]. \quad (3)$$

The transmittivity $T(k) = |t(k)|^2$ and the reflectivity $R(k) = |r(k)|^2$. We have re-derived (2,3) to find that for (1)

$$t_{left}(k) = t_{A,B}(k), r_{left}(k) = r_{A,B}(k) \quad (4)$$

$$\text{and } t_{right}(k) = t_{A,-B}(k), r_{right}(k) = r_{A,-B}(k).$$

Making multiple use of the property of Gamma functions namely $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}\pi z$ we express the transmitivity, $T(k)$ as

$$T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{(\sinh^2 \pi k + \sin^2 \pi A)(\sinh^2 \pi k + \cos^2 \pi B)} \quad (5)$$

Ordinarily, for real values of the parameters A, B the Eqs. (2-5) yield to the rule [4,8,9,11,15] of left/right handedness (non-reciprocity) of $R(k)$ and the general notion of non-unitarity. For other (in)variances see [14]. When $A = -(n+1) - i\alpha$ and $B = i\alpha - (n+1/2)$ with $n \in I^+ + \{0\}, \alpha > 0$ a recent phenomenon of spectral singularity [11] is observed wherein at $E = \alpha^2$ [14] both R and T become infinite. For other spacial values of the parameters A and B , from Eqs. (2-5) the following extra-ordinary features arise:

{1} Reciprocity and unitarity

Case 1:

When $A = n + 1/2, n \in I + \{0\}$ and B is real, from (2) and (3) we get

$$R(k) = \frac{\cos^2 \pi B}{\sinh^2 \pi k + \cos^2 \pi B}, T(k) = \frac{\sinh^2 \pi k}{\sinh^2 \pi k + \cos^2 \pi B}, \quad (6)$$

Case 2:

When $B \in I$ and A is real, we get

$$R(k) = \frac{\sin^2 \pi A}{\sinh^2 \pi k + \sin^2 \pi A}, T(k) = \frac{\sinh^2 \pi k}{\sinh^2 \pi k + \sin^2 \pi A}, \quad (7)$$

In both the cases from Eq.(4) the acclaimed reciprocity of reflectivity follows: $R_{left}(k) = R_{right}(k)$. The reflectivity is also symmetric under time-reversal: $R(-k) = R(k)$. The acclaimed unitarity can be checked readily using Eqs.(6,7).

{2} Invisibility with reciprocity:

In the above two cases of unitarity when $a = b = (n + 1/2)$ or $a = b = n$ check that two cases of invisibility occur wherein $R(k) = 0, T(k) = 1$ at any energy. This invisibility of the complex PT-symmetric potential (1) is not unidirectional,[15-17] it is from both left and right.

{3} Perfect transmission without reciprocity (of reflection) and unitarity:

When $A = B$, from Eqs.(2,3) we find $T(k) = 1$, but $R(k) \neq 0$. Unlike the case of invisibility here we have absence of : unitarity, reciprocity and time-reversal symmetry of the reflectivity. This phenomenon could be connected to the occurrence of crucial factors $e^{\pm 2k\epsilon}$ [5] in the reflectivity $R'(k) = e^{\pm 2k\epsilon} R(k)$ of a complex PT-symmetric scattering potential of the type $V_{PT}(x) = V_s(x + i\epsilon)$ [5] where $R(k)$ is the reflectivity of a real symmetric scattering potential. The transmittivity $T'(k)$ remains invariant and equals $T(k)$. If $T(k)$ of the real Hermitian potential $V_s(x)$ has transmission resonances ($T(k) = 1$) at real discrete energies, $V_{PT}(x)$ will have perfect transmission at these energies *sans* reciprocity and unitarity. It should be realized that V_{PT} (e.g., $\text{sech}(x + i\epsilon)$ [24]) when expanded in to its real and imaginary parts would mathematically appear to be formidably different from its real Hermitian primitive $V_s(x) = \text{sech}x$. We would like to remark that in a recent study [25] of Ginocchio potential [26,27], both the real one $V_G(x)$ and its PT-symmetric version $V_G(x + i\epsilon)$ [28] are the models of perfect transmission at discrete energies. The former is with reciprocity (of reflection) and unitarity and the latter without them.

To the best of our knowledge the above three paradoxical features $\{1-3\}$ in PT-symmetric structures are new and un-noticed so far. The proofs of the non-reciprocity of reflectivity in scattering form complex PT-symmetric potentials, to say the least, are incomplete and something is amiss there. The question whether there are other PT-symmetric structures yielding to the reciprocity (of reflectivity) and the unitarity is open for investigations. The present exposition provides a paradigm shift in the thinking in two ways. Firstly, we can have reciprocity and unitarity in scattering even without Hermiticity. Secondly, the PT-symmetric structures are more versatile than they are known to be so far.

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